Transparent Absorbing Boundary (TAB) for the Truncation of the Computational Domain

Jian Peng and Constantine A. Balanis

Abstract—**A new approach to domain truncation without reflection is proposed for finite methods. The open-space Maxwell's equations, along with boundary conditions, are transformed to an equivalent system with a homogeneous closed boundary; the latter is then solved numerically. Like the popular perfectly matched layer (PML), the new method is independent of frequency and incident angle. Its uniqueness is that it does not need the extra absorption region, since the field attenuation takes place in the domain of the subject.**

Index Terms— **Absorbing boundary, computational domain, truncation method, equivalent system.**

I. Introduction

TO TRUNCATE an unbounded space, a variety of techniques have been proposed [1]–[3]. The popular perfectly matched layer (PML) [4] provides a virtually reflection-free truncation that is independent of frequency and incident angle. However, an absorbing region, additional to the domain of interest, is needed in order for the outward-traveling wave to be absorbed.

The transparent absorbing boundary (TAB) proposed in this letter is a truncation method that forces the fields to decay *inside* the domain of interest and to become zero at the domain's boundary. The extra absorbing region used in the PML is thus eliminated. Since the method assumes no infinite geometry (i.e., half-space interfaces in PML or plane waves in Mur), it is possible to terminate a domain "conformally," which makes the computational domain even smaller. Due to its rigorous analytical approach, the method can be directly applied to various finite methods, such as finite difference and finite element, in either time or frequency domains.

II. Transparent Absorbing Boundary

Conventional absorption-based truncation methods focus on the design of absorbers (e.g., the loss mechanism and the geometry). However, the TAB method primarily emphasizes the results of absorption. The amplitudes of the fields are modulated in the computational domain to achieve reflection-free domain truncation.

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To initiate the formulation of the TAB, auxiliary fields $\mathbf{E}(t,r)$ and $\mathbf{H}(t,r)$ are first introduced and defined as

$$\mathbf{E}(t,r) = F(r)\mathbf{E}_o(t,r) \tag{1a}$$
$$\mathbf{H}(t,r) = F(r)\mathbf{H}_o(t,r) \tag{1b}$$

where $\mathbf{E}_o(t,r)$ and $\mathbf{H}_o(t,r)$ are the original physical fields of the problem of interest. $F(r)$ is a scalar function that decays outwardly and becomes zero at the boundary. An equivalence between the physical and auxiliary systems is then established by transforming Maxwell's equations for the physical fields $\mathbf{E}_o$ and $\mathbf{H}_o$, along with the boundary conditions, into the governing equations and boundary conditions for $\mathbf{E}$ and $\mathbf{H}$. Instead of solving Maxwell's equations in the unbounded space, one can first solve for the auxiliary fields in the finite closed domain. The physical fields interior to the boundary are then found with (1). Note that it is not necessary to find $\mathbf{E}_o$ and $\mathbf{H}_o$ on the boundary itself. The closed surface over which the equivalence principle is applied to find the far-zone $\mathbf{E}_o$ and $\mathbf{H}_o$ can be placed anywhere in the domain (as long as it is exterior to the subject of study); usually this is chosen at one or two cells interior to the truncation boundary.

Assume that r_o is the boundary of the subject domain, and $F(r) \neq 0$ for $r < r_o$. By expressing $(\mathbf{E}_o, \mathbf{H}_o)$ in terms of $(\mathbf{E}, \mathbf{H})$, one finds, from Maxwell's equations, the auxiliary system's governing equations for $r < r_o$ as

$$\frac{\partial \mathbf{E}}{\partial t} = \frac{1}{\epsilon}\left(\nabla \times \mathbf{H} - \frac{1}{F}\nabla F \times \mathbf{H}\right) - \frac{\sigma}{\epsilon}\mathbf{E} - \frac{F}{\epsilon}j_i \tag{2a}$$
$$\frac{\partial \mathbf{H}}{\partial t} = -\frac{1}{\mu}\left(\nabla \times \mathbf{E} - \frac{1}{F}\nabla F \times \mathbf{E}\right) - \frac{\sigma^*}{\mu}\mathbf{H} - \frac{F}{\mu}m_i \tag{2b}$$
$$\nabla \cdot \mathbf{E} = F\rho + \frac{1}{F}\nabla F \cdot \mathbf{E} \tag{2c}$$
$$\nabla \cdot \mathbf{H} = F\rho^* + \frac{1}{F}\nabla F \cdot \mathbf{H} \tag{2d}$$

where σ, σ^*, j_i, m_i, ρ and ρ^* retain their physical meanings. The most important feature of (2) is the introduction of the $\frac{1}{F}\nabla F \times \mathbf{E}$ and $\frac{1}{F}\nabla F \times \mathbf{H}$ terms that represent the losses in a system of hyperbolic partial differential equations [5], [6]. Regardless of their physical interpretations, it is these terms that result in the absorption of energy. Therefore, the TAB is an absorption-based technique, with the emphasis on the effect of the absorption.

On the boundary r_o, the auxiliary fields satisfy the following Dirichlet boundary conditions:

$$\mathbf{E} = F(r_o)\mathbf{E}_o \tag{3a}$$
$$\mathbf{H} = F(r_o)\mathbf{H}_o \tag{3b}$$

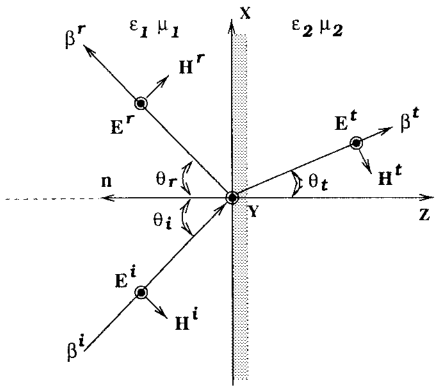

Fig. 1. TE (horizontal)-polarized plane wave incident at an oblique angle on an interface.

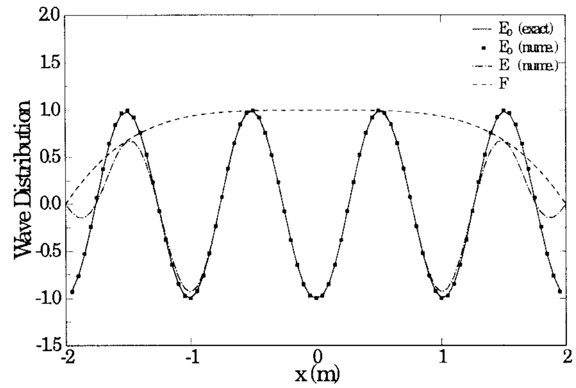

Fig. 2. The numerical solutions of the E_o of a plane wave traveling outwardly in both directions.

Apparently, homogeneous boundary conditions can be obtained for the auxiliary system if F is chosen properly. An example of such functions over a one-dimensional (1-D) domain is

$$F(x) = \left[1 - \left(\frac{|x|}{L_x}\right)^m\right]^n \tag{4}$$

where L_x is the length of attenuation path, while m and n should be no less than one.

To ensure that no artificial reflection is introduced during the transformation from the physical system to the auxiliary one, constraints must be imposed upon $F(r)$. It is well known that there will be no reflection from an interface if both the phase velocities and wave impedances are identical across it. Furthermore, the physical reflection at a medium discontinuity will not be affected by a superimposed artificial loss mechanism if the boundary conditions (BC), the phase velocities v, and the wave impedances η remain unchanged before and after the superposition; i.e.,

$$[\text{BC}]_a = [\text{BC}]_o, \quad v_a = v_o, \quad \text{and} \quad \eta_a = \eta_o \tag{5}$$

where subscript a indicates the auxiliary system while o represents the original one. Such a characteristic of zero reflection is independent of frequency, incident angle, geometry and medium. If F is *scalar* and *nonzero* at interior points $r < r_o$, the physical and auxiliary systems have identical wave impedances: $\eta_a(r) = \frac{|\mathbf{E}(t,r)|}{|\mathbf{H}(t,r)|} = \frac{|\mathbf{E}_o(t,r)|}{|\mathbf{H}_o(t,r)|} = \eta_o(r)$. Meanwhile, a *real* and *continuous* F guarantees identical boundary conditions and phase terms in the two systems. Therefore, no reflection is introduced during the transformation. In other words, the artificial loss seems *transparent*. Consequently, the proposed method is referred to as the *transparent absorbing boundary*.

III. Results

To demonstrate the concept of the TAB, the oblique incidence of a TE-polarized plane wave upon a half-space medium is examined analytically. The corresponding auxiliary fields are shown in Fig. 1. The incident "electric" field is in the form of

$$\mathbf{E}^i = \hat{a}_y|E^i(x,z)|e^{-j\frac{\omega}{v_1}(x\sin\theta_i + z\cos\theta_i)} \tag{6}$$

whose amplitude is no longer a constant. Other auxiliary fields can be expressed accordingly. The derivation is similar to the plane wave case in [7]. The continuity boundary condition derived from (5) leads to Snell's laws of reflection and transmission for the auxiliary fields

$$\theta_i = \theta_r \quad \text{and} \quad \frac{1}{v_1}\sin\theta_i = \frac{1}{v_2}\sin\theta_t \tag{7}$$

as well as the reflection and transmission coefficients Γ and T

$$\Gamma = \frac{\eta_2\cos\theta_i - \eta_1\cos\theta_t}{\eta_2\cos\theta_i + \eta_1\cos\theta_t} \quad \text{and} \quad T = \frac{2\eta_2\cos\theta_i}{\eta_2\cos\theta_i + \eta_1\cos\theta_t} \tag{8}$$

where $\eta_1 = \frac{|E^i|}{|H^i|}$ and $\eta_2 = \frac{|E^t|}{|H^t|}$ are the wave impedances of the auxiliary fields in medium 1 and 2, respectively. Since the phase velocities (hence, the propagation direction governed by Snell's law) and the wave impedances are identical in the physical and auxiliary systems, the coefficients given in (8) are equal to those of the physical case given in [7]. Therefore, the *transparent absorbing boundary* does not introduce additional reflections, regardless of the frequency and incident angle of the waves.

The absorption in the TAB is shown numerically as follows. Assume that a source is located in the middle of a one-dimensional unbounded space, and the plane wave propagates in both directions. Function (4) with $m = 4$ and $n = 1$ is used to terminate the computational domain at $|x| = 2$ (i.e., $L_x = 2$ m). Equations (2a) and (2b) in one dimension are approximated with the Lax–Wendroff scheme [8]. The Courant number $\gamma = \frac{c\delta t}{\Delta}$ is chosen to be one, the cell size is 0.025λ, and the time duration is 400 steps, which is sufficiently long for the waves to bounce back and forth a few times in the domain. The results are shown in Fig. 2. Dictated by the given F (the dashed line), the magnitude of the auxiliary field E (the dot–dashed line) decays outwardly and becomes zero at the truncation boundary. The physical field E_o is then found, using (1a), at all the grid points except the boundary ones where $F = 0$. The numerical and analytical solutions of E_o agree well, as indicated in the figure. It should be pointed out that the singularity of $1/F$ at the boundary poses no problem in the numerical implementation of (1a) and (1b). It is because

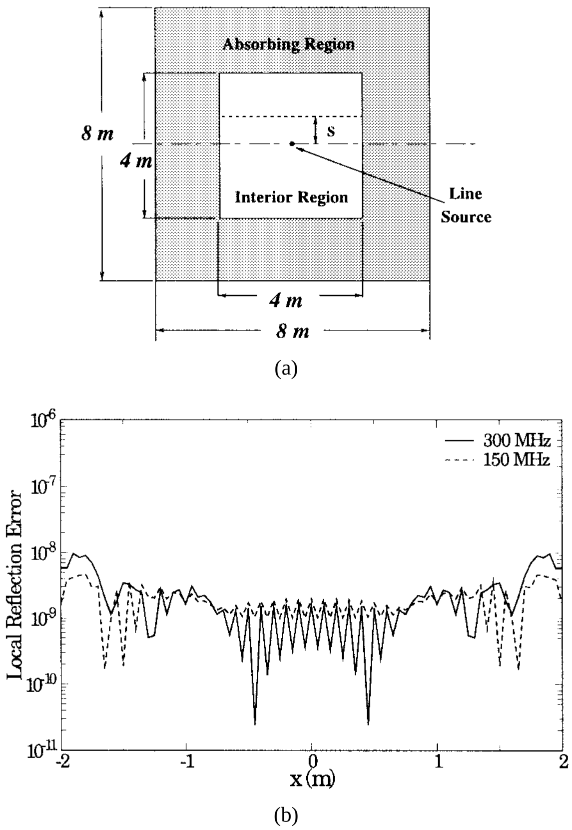

Fig. 3. Isolate and measure reflection errors due to the TAB only. (a) Testing setup. (b) Reflection error along the line $s = 1$ m.

the finite difference equations are set up only at the interior grids where $F \neq 0$.

In the above example, the reflection error of the TAB are inseparable from the error due to truncated high-order terms of the finite-difference scheme. To examine the reflection characteristics of the TAB numerically, the two-dimensional function

$$F(x,y) = \left[1 - \left(\frac{|x|-2}{2}\right)^4\right]\left[1 - \left(\frac{|y|-2}{2}\right)^4\right] \quad (9)$$

is defined only for the absorbing region, shown in Fig. 3(a), whose thickness is chosen to be equal to the half dimension of the interior region. Such a configuration allows the reflection from the TAB layer to enter deep into the interior region, before the exterior boundary's reflection (if any) reaches the interior boundary. Consequently, the computed reflection is *solely* due to the artificial loss mechanism in the TAB. A line source with TM polarization is placed in the center of the domain. The local reflection errors are computed in the interior region, using the methodology suggested by Moore [9]. Yee's algorithm [10] is used to approximate (2). The cells are 0.05 m $\times\, 0.05$ m. The Courant number $\gamma = c\delta t/\Delta$ is taken to be 0.7, and the time duration is 100 steps. Double precision was used. The total reflections from the four walls, with various incident angles and two frequencies, were collected along the dashed line in the interior region; and the results are shown in Fig. 3(b). As predicted, the computed numerical reflections are very low (i.e., 10^{-8} or -160 dB). It should be emphasized that the testing setup is to isolate the reflection error caused by the TAB's loss mechanism. In practice, a decaying F is applied to the interior domain only, as shown in Fig. 2.

Though the TAB is analytically reflection free, its numerical implementation may introduce reflections. When Yee's algorithm is used to approximate (2), a numerical perfect magnetic conductor (NPMC) wall is created near the truncation boundary [11], [12]. The problem is associated only with spatially staggered finite-difference schemes, such as Yee's. Collocated finite-difference schemes like the Lax–Wendroff scheme display no NPMC problem, as illustrated in Fig. 2.

IV. Conclusion

A new analytical approach, the *Transparent Absorbing Boundary* (TAB), has been proposed. The TAB introduces an artificial loss mechanism that can be mathematically identified. With the TAB method, a physical problem in an unbounded space can be solved in a finite closed domain, with the aid of the auxiliary fields.

Like the popular PML method, the TAB is reflection-free, independent of frequency, and unconstrained by the incident angle. The uniqueness of the TAB is that it does not need the additional absorbing region that is commonly used in the PML. Besides, the reflection characteristics (i.e., the *transparency*) of the TAB are independent of the curvature and corners of a computational domain, and it can be applied to truncate a domain that is "conformal" to the shape of the structure. Also, the simple analytical formulation of the TAB enables it to be applied directly in time or frequency domains.

References

[1] G. Mur, "Absorbing boundary conditions for the finite-difference approximation of the time-domain electromagnetic field equations," *IEEE Trans. Electromagn. Compat.*, vol. 23, pp. 377–382, 1981.

[2] Z. P. Liao, H. L. Wong, B. P. Yang, and Y. F. Yuan, "A transmitting boundary for transient wave analyzes," *Scientia Sinica* (series A), vol. XXVII, pp. 1063–1076, 1984.

[3] K. K. Mei and J. Fang, "Superabsorption—A method to improve absorbing boundary conditions," *IEEE Trans. Antennas Propagat.*, vol. 40, pp. 1001–1010, 1992.

[4] J. P. Berenger, "A perfectly matched layer for the absorption of electromagnetic waves," *J. Computational Phys.*, vol. 114, no. 2, pp. 185–200, Oct. 1994.

[5] P. R. Garabedian, *Partial Differential Equations*. New York: Wiley, 1964.

[6] F. Mainardi and D. Cocci, "Energy propagation in linear hyperbolic systems in the presence of dissipation," in *Proc. Fourth Int. Conf. on Hyperbolic Problems*, Taormina, Italy, Apr. 1992, pp. 409–415.

[7] C. A. Balanis, *Advanced Engineering Electromagnetics*. New York: Wiley, 1989.

[8] P. D. Lax and B. Wendroff, "Systems of conservation laws," *Comm. Pure and Appl. Math.*, vol. 13, pp. 217–237, 1960.

[9] T. G. Moore *et al.*, "Theory and application of radiation boundary operators," *IEEE Antennas. Propagat.*, vol. 36, pp. 1797–1812, Dec. 1988.

[10] K. S. Yee, "Numerical solution of initial boundary value problems involving Maxwell's equations in isotropic media," *IEEE Trans. Antennas Propagat.*, vol. AP-14, pp. 302–307, 1966.

[11] J. Peng and C. A. Balanis, "A new reflection-free truncation in finite methods: Transparent absorbing technique (TAT)," in *IEEE Antennas and Propagat. Soc. Int. Symp. 1996*, Baltimore, MD, July 21–26, 1996, vol. 1, pp. 88–91.

[12] J. Peng and C. A. Balanis, "Transparent absorbing boundary (TAB): Truncation of computational domain without reflections," Applied Comput. Electromag. Soc. (ACES) 13th Annu. Progress Rev., Monterey, CA, Mar. 17–21, 1997, submitted for publication.